d'Alembert solution

$a=4$, infinite string, $g(x)=0$ (initial velocity)

$f(x)=e^{-x^2}$, find $y(1,2)$

$$y(x,t)=\frac{1}{2}\left[f(x-at)+f(x+at)\right]+\frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds$$

$$=\frac{1}{2}\left[e^{-(x-4t)^2}+e^{-(x+4t)^2}\right]$$

Sturm-Liouville (orthogonality)

$y''+\lambda y=0 \qquad a<x<b$

$\alpha_1 y(a)+\alpha_2 y'(a)=0$

$\beta_1 y(b)+\beta_2 y'(b)=0$

solve for y_n for different cases of λ ($\lambda<0$, $\lambda=0$, $\lambda>0$)

the y_n are mutually orthogonal $\rightarrow$ $\int_a^b y_n y_m \, dx = 0$ if $n \neq m$

they can be used to expand some function $f(x)$

$$f(x) = \sum_{n=1}^{\infty} c_n y_n \qquad \text{(just like w/ Fourier series)}$$

to find c_n, we use orthogonality of y_n

multiply by y_m

$$f(x) y_m = \sum_{n=1}^{\infty} c_n y_n y_m$$

$$= c_1 y_1 y_m + c_2 y_2 y_m + \cdots$$

integrate over $a < x < b$

$$\int_a^b f(x) y_m \, dx = \underbrace{\int_a^b c_1 y_1 y_m \, dx + \int_a^b c_1 y_2 y_m \, dx + \cdots}_{\text{all zero except when } n=m}$$

$$\int_a^b f(x)\, y_n\, dx = \int_a^b C_n (y_n)^2\, dx \to C_n = \frac{\int_a^b f(x)\, y_n\, dx}{\int_a^b (y_n)^2\, dx}$$

$$= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

or $\cos\left(\frac{n\pi x}{L}\right)$ if

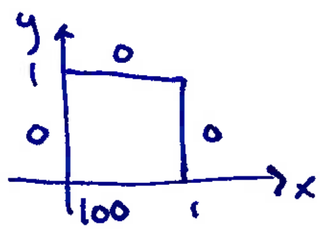
$$y_n = \sin\left(\frac{n\pi x}{L}\right) \text{ or } \cos\left(\frac{n\pi x}{L}\right)$$

2-D Heat

$$u_t = k(u_{xx} + u_{yy}) \qquad 0 < x < 1 \qquad 0 < y < 1$$

~~u(x, 0) = 100~~

ex:



$u(x, 0) = 100$

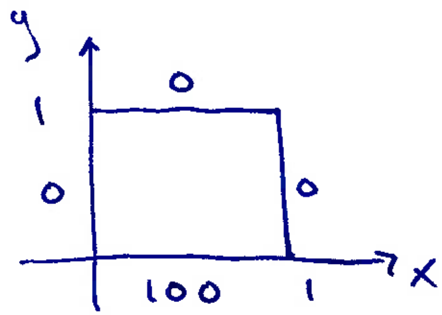
$u(x, 1) = 0$

$u(0, y) = 0$

$u(1, y) = 0$

$$u(x, y) = \sum_{n=1}^{\infty} C_n \sin(n\pi x) \sinh(n\pi(1 - y))$$

Solution has no time → Steady state



$$u_{xx} + u_{yy} = 0$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$X'' + \lambda X = 0 \qquad X(0) = X(1) = 0$$

$$\lambda_n = n^2\pi^2 \qquad X_n = \sin(n\pi x)$$

$$Y'' - \lambda Y = 0$$

$$Y'' - n^2\pi^2 Y = 0 \qquad Y(1) = 0 \rightarrow Y(z=0) = 0$$

$$Y = A\cosh(n\pi y) + B\sinh(n\pi y)$$

let $\cancel{z = 1-y}$ $\quad y = 1 - z \qquad z = 1 - y$

$$Y(z) = A\cosh(n\pi(1-z)) + B\sinh(n\pi(1-z)) \rightarrow \cdots \rightarrow$$ Sinh survives

$$u(x,y) = \sum C_n \sin(n\pi x)\sinh(n\pi(1-y))$$

$$u(x,0) = 100 = \sum_{n=1}^{\infty} C_n \sin(n\pi x)\sinh(n\pi)$$

$$100 = \sum_{n=1}^{\infty} \left[C_n \sinh(n\pi)\right] \sin(n\pi x) \qquad \text{sine series } L=1$$

$$C_n \sinh(n\pi) = \frac{2}{1}\int_0^1 100 \sin(n\pi x)\,dx$$

$$C_n = \frac{2}{\sinh(n\pi)}\int_0^1 100\sin(n\pi x)\,dx$$

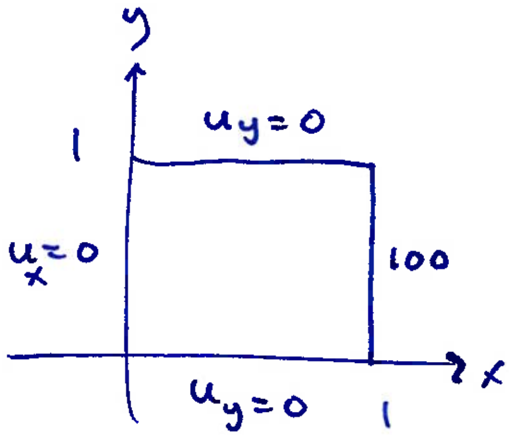
if not steady-state: $u(x,y,t) = \sum C_n T_{nm} X_n Y_m$

exponential vs hyperbolic

$\downarrow$ dealing w/ ∞

$\hookrightarrow$ finite domain

$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$



$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

solve Y first: want $Y'' + \lambda Y = 0$

$$Y'' + \lambda Y = 0 \qquad Y'(0) = Y'(1) = 0$$

$$\lambda_n = \frac{n^2\pi^2}{1^2} = n^2\pi^2$$

$$Y_n = \cos(n\pi y) \qquad n = 0, 1, 2, 3, \ldots$$

$$X'' - \lambda X = 0$$

$$X'' - n^2\pi^2 X = 0$$

$$X = A\cosh(n\pi x) + B\sinh(n\pi x)$$

$$X'(0) = 0$$

$$X' = n\pi A\sinh(n\pi x) + n\pi B\cosh(n\pi x)$$

$$0 = n\pi B \rightarrow B = 0$$

$$X_n = \cosh(n\pi x)$$